International Journal of Theoretical Physics, Vol. 40, No. 9, September 2001 (© 2001)

Theory of Super Phase-Space Representations and Supercoherent States

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Received August 21, 2000

The theory of a class of phase-space representations is developed for boson–fermion systems. The super phase-space operator is constructed and its properties are discussed. It is shown that the supersymmetric antinormal ordering rule corresponds to the supercoherent-state representation. Thus, the supersymmetric nature of the supercoherent states is revealed from the viewpoint of the phase-space representations.

1. INTRODUCTION

Method of phase-space representations has been attracting continuous interest in quantum physics (Agarwal and Wolf, 1970; Balazs and Jennings, 1984; Hillery *et al.*, 1984; Kim and Noz, 1991; Scully and Zubairy, 1997). It allows to express the quantum expectation value of an observable as the statistical average of a corresponding physical quantity with respect to a certain distribution function in quantum phase-space. The celebrated examples are the Wigner distribution function, the Sudarshan–Glauber P-distribution function, and the Husimi Q-distribution function, which correspond to the Weyl ordering, the normal ordering, and the antinormal ordering of the product of the bosonic creation and annihilation operators, respectively.

Compared with the long tradition of the studies of phase-space representations of bosons, fermionic theory is relatively new. It was about a decade ago that the Wigner distribution function of fermions was constructed in Abe and Suzuki (1989) and applied to the optical Dicke model in Abe and Suzuki (1992). The discussion was further generalized to the case of supersymmetric systems in Abe (1992), where the super Wigner function was defined and its properties were analyzed. In these works, only the Wigner distribution function and the associated Weyl ordering rule were considered. Therefore, a question remains regarding the existence of a class of representations including the P- and Q-distribution functions.

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0020-7748/01/0900-1647$19.50/0

In this paper, we establish a class of phase-space representations of boson–fermion systems. We construct the phase-space operator which yields quantum-classical correspondence in super phase-space. Then, we discuss its invariance under supersymmetry transformations. It is shown that the phase-space operator is supersymmetric if and only if the ordering rules for boson and fermion are identical. We also show that the phase-space representation with the antinormal ordering is equivalent to the supercoherent-state representation. Thus, the supersymmetric nature of the supercoherent state is revealed from the phase-space approach. Throughout this paper, single-mode bosonic and fermionic fields are treated for simplicity, but extension of the whole discussion to the case of multimode fields is straightforward.

2. ONE-PARAMETER FAMILY OF FERMIONIC PHASE-SPACE OPERATORS

It seems appropriate to begin this section with a brief summary of the theory of phase-space representations of the boson. After it, we shall discuss the fermionic theory in correspondence to the bosonic theory.

The creation and annihilation operators of the boson, $\hat{a}^\dagger$ and $\hat{a}$, satisfy the following commutation relations:

$$[\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \tag{1}$$

The bosonic phase-space operator is defined by Agarwal and Wolf (1970)

$$\hat{\Delta}_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 z \exp\left[-\frac{s_{\rm b}}{2} z^* z + (\hat{a}^\dagger - \alpha^*) z - z^*(\hat{a} - \alpha) \right]. \tag{2}$$

Here, z and α are ordinary complex variables and $d^2 z \equiv d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)$. In particular, α and α^* label the complex classical phase space. $s_{\rm b}$ is a real parameter. This operator satisfies the relation

$$\mathrm{Tr}\left[\hat{\Delta}_{\rm b}^{(s_{\rm b})}(\alpha_1, \alpha_1^*) \hat{\Delta}_{\rm b}^{(-s_{\rm b})}(\alpha_2, \alpha_2^*) \right] = \frac{1}{\pi} \delta^{(2)}(\alpha_1 - \alpha_2). \tag{3}$$

The phase-space representation of the density operator $\hat{\rho}_{\rm b}$ of the bosonic system is given by

$$F_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*) = \mathrm{Tr}\left[\hat{\rho}_{\rm b} \hat{\Delta}_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*) \right]. \tag{4}$$

Using the identical relation, $\mathrm{Tr}(\hat{\rho}_{\rm b}) = 1$, and the complex Fourier transformation of the delta function

$$\frac{1}{\pi^2} \int d^2 \alpha \exp[\pm(\alpha^* z - z^* \alpha)] = \delta^{(2)}(z) \equiv \delta(\mathrm{Re}\, z) \delta(\mathrm{Im}\, z), \tag{5}$$

we see that the phase-space function in Eq. (4) is normalized:

$$\int d^2\alpha F_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*) = 1. \tag{6}$$

The phase-space operator in Eq. (2) gives the correspondence relation between the quantum operator and its classical counterpart in phase space:

$$\hat{Q}_{s_{\rm b}}(\hat{a}, \hat{a}^\dagger) = \int d^2\alpha\, Q(\alpha, \alpha^*)\hat{\Delta}_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*), \tag{7}$$

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\mathrm{Tr}\big[(\hat{Q}_{s_{\rm b}}(\hat{a}, \hat{a}^\dagger)\hat{\Delta}_{\rm b}^{(-s_{\rm b})}(\alpha, \alpha^*)\big], \tag{8}$$

where $\hat{Q}_{s_{\rm b}}(\hat{a}, \hat{a}^\dagger)$ is the $s_{\rm b}$-ordered operator obtained by quantizing the classical quantity $Q(\alpha, \alpha^*)$ which is a polynomial function of α and α^*. The three cases, $s_{\rm b} = 0, \pm 1$, are particularly important: $s_{\rm b} = 0$, $s_{\rm b} = -1$, and $s_{\rm b} = +1$ correspond to the Weyl ordering, the normal ordering, and the antinormal ordering, respectively. For example,

$$(\hat{a}^\dagger\hat{a})_{\rm W} = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \int d^2\alpha\, \alpha^*\alpha\, \hat{\Delta}_{\rm b}^{(0)}(\alpha, \alpha^*), \tag{9}$$

$$(\hat{a}^\dagger\hat{a})_{\rm N} = \hat{a}^\dagger\hat{a} = \int d^2\alpha\, \alpha^*\alpha\, \hat{\Delta}_{\rm b}^{(-1)}(\alpha, \alpha^*), \tag{10}$$

$$(\hat{a}^\dagger\hat{a})_{\rm AN} = \hat{a}\hat{a}^\dagger = \int d^2\alpha\, \alpha^*\alpha\, \hat{\Delta}_{\rm b}^{(+1)}(\alpha, \alpha^*). \tag{11}$$

Correspondingly, $F_{\rm b}^{(0)}(\alpha, \alpha^*)$, $F_{\rm b}^{(-1)}(\alpha, \alpha^*)$, and $F_{\rm b}^{(+1)}(\alpha, \alpha^*)$ are the Wigner distribution function, the Sudarshan–Glauber P-distribution function, and the Husimi Q-distribution function, respectively. The quantum expectation value of the $s_{\rm b}$-ordered physical quantity in Eq. (7) can be expressed as the phase-space average of the corresponding classical quantity with respect to $F_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*)$:

$$\mathrm{Tr}[\hat{\rho}_{\rm b}\hat{Q}_{s_{\rm b}}(\hat{a}, \hat{a}^\dagger)] = \int d^2\alpha\, Q(\alpha, \alpha^*)F_{\rm b}^{(s_{\rm b})}(\alpha, \alpha^*). \tag{12}$$

Now, we wish to develop the theory of the phase-space representations of the fermion. The fermionic creation and annihilation operators, $\hat{b}^\dagger$ and $\hat{b}$, satisfy the anticommutation relations

$$\{\hat{b}, \hat{b}^\dagger\} = 1, \qquad \{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0. \tag{13}$$

Let us define the following fermionic phase-space operator:

$$\hat{\Delta}_{\rm f}^{(s_{\rm f})}(\beta, \beta^*) = \int d^2\zeta\, \exp\Big[-\frac{s_{\rm f}}{2}\zeta^*\zeta + (\hat{b}^\dagger - \beta^*)\zeta - \zeta^*(\hat{b} - \beta)\Big]. \tag{14}$$

In this equation, ζ, ζ^*, β, and β^* are the anticommuting classical variables called the Grassmann-odd variables, or simply the G-odd variables. $\zeta\zeta^*$ is G-even. For example, $\zeta^2 = \zeta^{*2} = \zeta\zeta^* + \zeta^*\zeta = 0$, $(\zeta\beta^*)^* = \beta\zeta^* = -\zeta^*\beta$, $(\hat{b}\zeta)^\dagger = \zeta^*\hat{b}^\dagger = -\hat{b}^\dagger\zeta^*$, and so on. The G-integrations are normalized as follows:

$$\int d^2\zeta\,\zeta\zeta^* = 1, \qquad \int d^2\zeta\,\zeta = \int d^2\zeta\,\zeta^* = 0, \qquad \int d^2\zeta = 0. \tag{15}$$

It can be found that the phase-space operator in Eq. (14) satisfies the relation

$$\mathrm{Tr_g}\big[\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta_1, \beta_1^*)\,\hat{\Delta}_{\mathrm{f}}^{(-s_\mathrm{f})}(\beta_2, \beta_2^*)\big] = \delta^{(2)}(\beta_1 - \beta_2), \tag{16}$$

where $\delta^{(2)}(\beta)$ is the G-delta function

$$\delta^{(2)}(\beta) \equiv \beta\,\beta^* = \int d^2\zeta\,\exp[\pm(\zeta^*\beta - \beta^*\zeta)]. \tag{17}$$

An important point is the fact that $\mathrm{Tr_g}$ in this equation is not the ordinary trace operation but the graded trace operation, which is, in the Fock representation, defined by

$$\mathrm{Tr_g}(\hat{Q}) = \sum_{n=0,1} (-)^n \langle n|\hat{Q}|n\rangle, \tag{18}$$

where $\{|0\rangle, |1\rangle = \hat{b}^\dagger|0\rangle\}$ is the Fock basis. The ground state $|0\rangle$ is assumed to be G-even, that is, $\beta|0\rangle = |0\rangle\beta$ with G-odd β. The graded trace operation has the cyclicity property for two special cases:[2] $\mathrm{Tr_g}(\hat{X}\hat{Y}) = \mathrm{Tr_g}(\hat{Y}\hat{X})$ if both $\hat{X}$ and $\hat{Y}$ are G-even, $\mathrm{Tr_g}(\hat{X}\hat{Y}) = -\mathrm{Tr_g}(\hat{Y}\hat{X})$ if both $\hat{X}$ and $\hat{Y}$ are G-odd. Like the bosonic phase-space operator, $\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta, \beta^*)$ defines the correspondence relation between the basic quantum operators and the classical phase-space variables:

$$\hat{b} = \int d^2\beta\,\beta\,\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta, \beta^*), \qquad \mathrm{Tr_g}\big[\hat{b}\,\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta, \beta^*)\big] = \beta, \tag{19}$$

$$\hat{b}^\dagger = \int d^2\beta\,\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta, \beta^*)\beta^*, \qquad \mathrm{Tr_g}\big[\hat{\Delta}_{\mathrm{f}}^{(s_\mathrm{f})}(\beta, \beta^*)\,\hat{b}^\dagger\big] = \beta^*. \tag{20}$$

s_f in Eq. (14) is a real parameter, which is responsible for the operator ordering rule, analogously to the bosonic theory. The cases $s_\mathrm{f} = 0$, $s_\mathrm{f} = -1$, and $s_\mathrm{f} = +1$ correspond to the Weyl ordering, the normal ordering, and the antinormal ordering, respectively. For example,

$$(\hat{b}^\dagger\hat{b})_\mathrm{W} = \frac{1}{2}(\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger) = \int d^2\beta\,\beta^*\beta\,\hat{\Delta}_{\mathrm{f}}^{(0)}(\beta, \beta^*), \tag{21}$$

[2] The statements made in Abe (1992) and Abe and Suzuki (1989, 1992) regarding the cyclicity property of the graded trace operation should be corrected as follows. $\mathrm{Tr_g}(\hat{X}\hat{Y}) = \pm\mathrm{Tr_g}(\hat{X}\hat{Y})$, where the sign $-(+)$ is taken for the case when both $\hat{X}$ and $\hat{Y}$ are G-odd (G-even). However, there does not exist the cyclicity property when $\hat{X}$ and $\hat{Y}$ are statistically different from each other (e.g., when $\hat{X}$ is G-even whereas $\hat{Y}$ is G-odd).

$$(\hat{b}^{\dagger}\hat{b})_{\mathrm{N}} = \hat{b}^{\dagger}\hat{b} = \int d^2\beta\, \beta^*\beta\, \hat{\Delta}_{\mathrm{f}}^{(-1)}(\beta, \beta^*), \tag{22}$$

$$(\hat{b}^{\dagger}\hat{b})_{\mathrm{AN}} = -\hat{b}\hat{b}^{\dagger} = \int d^2\beta\, \beta^*\beta\, \hat{\Delta}_{\mathrm{f}}^{(+1)}(\beta, \beta^*). \tag{23}$$

In the earlier discussions (Abe, 1992; Abe and Suzuki, 1989, 1992), the Wigner distribution function of the fermion has been constructed using the graded trace operation. In that case, the quantum expectation value of the Weyl-ordered operator $\hat{Q}_W(\hat{b}, \hat{b}^{\dagger})$ is expressed as the phase-space average of the G-Fourier transformation of the corresponding classical quantity $Q(\beta, \beta^*)$ with respect to the Wigner distribution function, not as the phase-space average of $Q(\beta, \beta^*)$ itself. Here, we wish to examine another possibility: we propose to define the phase-space distribution function using the ordinary trace operation. That is,

$$F_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*) \equiv \mathrm{Tr}\big[\hat{\rho}_{\mathrm{f}}\hat{\Delta}_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*)\big], \tag{24}$$

where $\hat{\rho}_{\mathrm{f}}$ is the density operator of the fermionic system. There are two advantageous points in this definition. First of all, in contrast to the earlier definition using the graded trace operation, the normalization condition $\int d^2\beta\, F_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*) = 1$ is immediately fulfilled, because $\int d^2\beta\, \hat{\Delta}_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*) = 1$ and $\mathrm{Tr}(\hat{\rho}_{\mathrm{f}}) = 1$. Second, the quantum expectation value of the s_{f}-ordered quantity

$$\hat{Q}_{s_{\mathrm{f}}}(\hat{b}, \hat{b}^{\dagger}) = \int d^2\beta\, Q(\beta, \beta^*)\hat{\Delta}_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*) \tag{25}$$

can be expressed directly as the phase-space average of $Q(\beta, \beta^*)$ with respect to $F_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*)$:

$$\mathrm{Tr}[\hat{\rho}_{\mathrm{f}}\hat{Q}_{s_{\mathrm{f}}}(\hat{b}, \hat{b}^{\dagger})] = \int d^2\beta\, Q(\beta, \beta^*)F_{\mathrm{f}}^{(s_{\mathrm{f}})}(\beta, \beta^*). \tag{26}$$

As an example, let us consider a single fermion at finite temperature. The canonical density operator is given by $\hat{\rho}_{\mathrm{f}} = \exp(-\hat{H}_W)/\mathrm{Tr}[\exp(-\hat{H}_W)]$, where $\hat{H}_W(\hat{b}, \hat{b}^{\dagger}) = \hat{b}^{\dagger}\hat{b} - 1/2$ is the system Hamiltonian ($\hbar\omega = k_B T \equiv 1$). This form of the Hamiltonian corresponds to the Weyl ordering. The classical counterpart of $\hat{H}_W(\hat{b}, \hat{b}^{\dagger})$ is $H(\beta, \beta^*) = \beta^*\beta$. The associated Wigner function is calculated to be $F_{\mathrm{f}}^{(0)}(\beta, \beta^*) = \mathrm{Tr}[\hat{\rho}_{\mathrm{f}}\hat{\Delta}_{\mathrm{f}}^{(0)}(\beta, \beta^*)] = \beta\beta^* + (1/2)\tanh(1/2)$. Then, the familiar result is obtained for the internal energy: $\langle \hat{H}_W \rangle = \int d^2\beta\, H(\beta, \beta^*)F_{\mathrm{f}}^{(0)}(\beta, \beta^*) = -(1/2)\tanh(1/2)$.

Thus, we see the complete parallelism between the phase-space representations of the s_{b}-ordered bosonic theory and the s_{f}-ordered fermionic theory. This parallelism becomes essential when the supersymmetric systems are treated.

3. SUPERSYMMETRIC-ORDERED THEORY

Consider a system composed of the boson and the fermion. The total phase-space operator is given by the product of the bosonic and fermionic phase-space operators:

$$\hat{\Delta}_{\mathrm{b,f}}^{(s_{\mathrm{b}},s_{\mathrm{f}})}(\alpha,\alpha^*,\beta,\beta^*) = \frac{1}{\pi^2}\iint d^2z\, d^2\zeta\, \exp\left(-\frac{s_{\mathrm{b}}}{2}z^*z - \frac{s_{\mathrm{f}}}{2}\zeta^*\zeta + \hat{A}^\dagger z - z^*\hat{A} + \hat{B}^\dagger\zeta - \zeta^*\hat{B}\right), \tag{27}$$

where $\hat{A}^\dagger$, $\hat{A}$, $\hat{B}^\dagger$, and $\hat{B}$ are the displaced operators

$$\hat{A} = \hat{a} - \alpha, \qquad \hat{A}^\dagger = \hat{a}^\dagger - \alpha^*, \tag{28}$$

$$\hat{B} = \hat{b} - \beta, \qquad \hat{B}^\dagger = \hat{b}^\dagger - \beta^*. \tag{29}$$

Now, let us examine the supersymmetry transformation of the displaced operators as follows:

$$\hat{A} \to \hat{A} + \hat{B}\theta, \qquad \hat{A}^\dagger \to \hat{A}^\dagger + \theta\hat{B}^\dagger, \tag{30}$$

$$\hat{B} \to \hat{B} + \hat{A}\theta, \qquad \hat{B}^\dagger \to \hat{B}^\dagger + \theta\hat{A}^\dagger. \tag{31}$$

where θ is a nilpotent real G-odd variable, i.e., $\theta\,\hat{B} = -\hat{B}\,\theta$ and $\theta^2 = 0$, for example. This transformation is generated by the Hermitian operator

$$\hat{G} = \hat{A}\hat{B}^\dagger + \hat{B}\hat{A}^\dagger. \tag{32}$$

In fact, we see

$$[\theta\hat{G}, \hat{A}] = \hat{B}\theta, \qquad [\theta\hat{G}, \hat{A}^\dagger] = \theta\hat{B}^\dagger, \tag{33}$$

$$[\theta\hat{G}, \hat{B}] = \hat{A}\theta, \qquad [\theta\hat{G}, \hat{B}^\dagger] = \theta\hat{A}^\dagger. \tag{34}$$

An important point is that the above transformation can be compensated by the following change of the integration variables:

$$z \to z + \zeta\theta, \qquad z^* \to z^* + \theta\zeta^*, \tag{35}$$

$$\zeta \to \zeta + z\theta, \qquad \zeta^* \to \zeta^* + \theta z^*. \tag{36}$$

if and only if

$$s_{\mathrm{b}} = s_{\mathrm{f}}. \tag{37}$$

In addition, the graded Jacobian factor defined in terms of the graded determinant (the superdeterminant) (Berezin, 1987) associated with Eqs. (35) and (36) is equal to unity:

$$\det_{g}\begin{pmatrix}1 & 0 & \theta & 0\\ 0 & 1 & 0 & \theta\\ \theta & 0 & 1 & 0\\ 0 & -\theta & 0 & 1\end{pmatrix} = \det\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\theta & 0\\ 0 & \theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}\theta & 0\\ 0 & -\theta\end{pmatrix}\right]$$

$$\times \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^{-1}$$

$$= 1, \tag{38}$$

and therefore the measure of integration $d^2z\, d^2\zeta$ in Eq. (27) remains invariant.

Thus, we conclude that

$$\hat{\Delta}^{(s)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*) = \frac{1}{\pi^2}\iint d^2z\, d^2\zeta \exp\left[-\frac{s}{2}(z^*z + \zeta^*\zeta) + \hat{A}^{\dagger}z - z^*\hat{A} + \hat{B}^{\dagger}\zeta - \zeta^*\hat{B}\right] \tag{39}$$

is the supersymmetric phase-space operator, which is referred to as the super phase-space operator. The super phase-space distribution function

$$F^{(s)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*) = \mathrm{Tr}\left[\hat{\rho}\,\hat{\Delta}^{(s)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*)\right] \tag{40}$$

is now regarded as a superfield. It admits the superfield expansion

$$F^{(s)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*) = f_0(\alpha, \alpha^*) + \beta f_1^*(\alpha, \alpha^*) + f_1(\alpha, \alpha^*)\beta^* + \beta\,\beta^* f_2(\alpha, \alpha^*), \tag{41}$$

where f_0 and f_2 are the real-valued functions and f_1 and f_1^* are complex G-odd functions.

4. SUPERCOHERENT STATES

Using the super phase-space operator in Eq. (39), we can obtain the supersymmetric generalization of a class of phase-space distribution functions including the Wigner distribution function ($s = 0$), the Sudarshan–Glauber P-distribution function ($s = -1$), and the Husimi Q-distribution function ($s = +1$). Here, of particular interest for us is the supersymmetric Q-representation, since it is expected to be connected directly with the supercoherent states.

The supercoherent states have repeatedly been discussed in the literature. Examples are found in Aragone and Zypman (1986), Fatyga *et al.* (1991), Jayaraman *et al.* (1999), and Kostelecky *et al.* (1993). Similarly to Fatyga *et al.* (1991) and Kostelecky *et al.* (1993), we define the supercoherent state as follows:

$$|\alpha, \alpha^*, \beta, \beta^*\rangle = \exp(\hat{a}^{\dagger}\alpha - \alpha^*\hat{a})\exp(\hat{b}^{\dagger}\beta - \beta^*\hat{b})|0, 0\rangle, \tag{42}$$

where $|0, 0\rangle$ is the product of the ground states of the boson and the fermion. α and α^* are the ordinary complex variables, whereas β and β^* are the complex G-odd variables. This is the simultaneous eigenstate of $\hat{a}$ and $\hat{b}$:

$$\hat{a}\ |\alpha, \alpha^*, \beta, \beta^*\rangle = \alpha\ |\alpha, \alpha^*, \beta, \beta^*\rangle, \tag{43}$$

$$\hat{b}\ |\alpha, \alpha^*, \beta, \beta^*\rangle = \beta\ |\alpha, \alpha^*, \beta, \beta^*\rangle. \tag{44}$$

It is a normalized state and possesses the (over) completeness

$$\frac{1}{\pi}\iint d^2\alpha\, d^2\beta\ |\alpha, \alpha^*, \beta, \beta^*\rangle\langle\alpha, \alpha^*, \beta, \beta^*| = 1. \tag{45}$$

The supercoherent state in Eq. (42) is simply the product of the bosonic and fermionic coherent states. Based only on this fact, one hardly sees its supersymmetric nature. We wish to clarify this point by showing that, in fact, the supercoherent state corresponds to $\hat{\Delta}^{(+1)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*)$, which is supersymmetric as shown in Section 3.

To see the explicit relation between the super phase-space operator $\hat{\Delta}^{(+1)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*)$ and the supercoherent state, let us employ the well-known formula, $\exp(\hat{X} + \hat{Y}) = \exp(\hat{X})\exp(\hat{Y} - \frac{1}{2}[\hat{X}, \hat{Y}])$ with $[\hat{X}, \hat{Y}]$ being a c number, and rewrite the operator as follows:

$$\hat{\Delta}^{(+1)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*) = \frac{1}{\pi^2}\iint d^2z\, d^2\zeta\ \exp(-z^*\hat{A} - \zeta^*\,\hat{B}) \times \exp(\hat{A}^\dagger z + \hat{B}^\dagger \zeta). \tag{46}$$

Inserting Eq. (45) between the two exponential operators in the integrand in Eq. (46) and using Eqs. (5) and (17), we find

$$\begin{aligned}
&\hat{\Delta}^{(+1)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*)\\
&= \frac{1}{\pi^3}\iint d^2z\, d^2\zeta \iint d^2\alpha'\, d^2\beta'\ \exp[(\alpha'^* - \alpha^*)z - z^*(\alpha' - \alpha)]\\
&\quad\times \exp[(\beta'^* - \beta^*)\zeta - \zeta^*(\beta' - \beta)]\ |\alpha', \alpha'^*, \beta', \beta'^*\rangle\langle\alpha', \alpha'^*, \beta', \beta'^*|\\
&= \frac{1}{\pi}|\alpha, \alpha^*, \beta, \beta^*\rangle\langle\alpha, \alpha^*, \beta, \beta^*|.
\end{aligned} \tag{47}$$

Therefore, as expected, $\hat{\Delta}^{(+1)}_{\mathrm{SUSY}}(\alpha, \alpha^*, \beta, \beta^*)$ corresponds to the supercoherent-state representation. Conversely, the supersymmetric aspect of the supercoherent state is thus clarified from the viewpoint of the theory of phase-space representations.

5. CONCLUSION

We have developed the theory of phase-space representations of the boson-fermion system. We have clarified the supersymmetric structure of the super phase-space operators associated with a class of the ordering rules. It has been shown that the supersymmetric antinormal ordering rule corresponds to the supercoherent-state representation. In this way, the supersymmetric nature of the supercoherent state is revealed.

ACKNOWLEDGMENT

This work was supported in part by the GAKUJUTSU-SHO Program of College of Science and Technology, Nihon University and by the Grant-in-Aid for Scientific Research of Japan Society for the Promotion of Science.

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